

# Homotopy analysis method for stochastic differential equations

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## Abstract

The homotopy analysis method known from its successful applications to obtain quasi-analytical approximations of solutions of ordinary and partial differential equations is applied to stochastic differential equations with Gaussian stochastic forces and to the Fokker-Planck equations. Only the simplest non-trivial examples of such equations are considered, but such that they can almost immediately be translated to those which appear in the stochastic quantization of a nonlinear scalar field theory. It has been found that the homotopy analysis method yields excellent agreement with exact results (when the latter are available) and appears to be a very promising approach in the calculations related to quantum field theory and quantum statistical mechanics.

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## 1. Introduction

The unprecedented recent growth of computing power of modern machines as well as the development of the software has made it possible to substantially enhance the quantitative predictive capabilities of physical research. Nevertheless, it appears that the role played by analytical techniques to solve mathematical problems of science and engineering has by no means diminished. And the development of symbolic algebra systems has enabled the

analytical methods to become more reliable and efficient. Among those analytical methods one can mention a particularly efficient approach recently developed by Liao [1, 2, 3, 4, 5, 6] called by that author the “homotopy analysis method”. One of the most significant features of the homotopy analysis method is its independence of the presence

of any small parameter. The approximate solution has the form of a series expansion. Usually, it is required that the series is truncated at a rather high order. However, skillful choice — e.g., on using variational tools — of some additional control parameters

(the presence of which the method admits) allows one to obtain meaningful results even in low orders, as reported by Marinca and coworkers [7, 8, 9].

The purpose of this paper is to demonstrate usefulness of the homotopy analysis method in the case of stochastic differential equations. To illustrate the quality of approximation provided by Liao’s method, we use a single and very simple stochastic differential equation, all statistical properties of its solution being easily obtained from the corresponding Fokker-Planck (Kolmogorov forward) equation. Our choice of working example has been, however, also motivated by the far-reaching analogy with the finite-temperature quantum field theory in the stochastic quantization framework.

The main body of this work is organized as follows. Section 2 contains a short description of the Liao as well as Marinca approaches to obtain series solutions of the differential equations. In Section 3 such a series is obtained for a stochastic differential equation related to the so-called “zero-dimensional” quantum field theory. Section 4 is devoted to the solution of the corresponding Fokker-Planck equation while Section 5 contains several final remarks.

## 2. Homotopy analysis method by Liao

Let us consider an arbitrary system of differential equations (linear or non-linear, ordinary or partial, homogeneous or not) of the form

$$\mathcal{N}_a(u, t) = 0 \tag{1}$$

where the index  $a$  enumerates the equations of the system,  $u$  is the vector of dependent variables, and  $t$  represents the set of independent variables. The family of solutions will of course not change if we multiply the left-hand side by a parameter  $q$ ,  $q \neq 0$ . Alongside (1) let us also consider a one-parameter family of systems of the form:

$$[(1 - q)\mathcal{L}(u(t) - u^{(0)}(t))]_a + q\mathcal{N}_a(u, t) = 0, \quad (2)$$

where  $\mathcal{L}$  is a linear operator such that  $\mathcal{L}u = 0$  can easily be solved while  $u^{(0)}$  is an initial guess of the solution. Very often  $u^{(0)}$  is chosen either to satisfy  $\mathcal{L}u^{(0)} = 0$  or simply  $u^{(0)} = 0$ . Thus, for  $q = 0$  we have to do with an easy to solve linear system while for  $q = 1$  we obtain the initial system. The idea of Liao has been to solve the system (2) by expanding  $u$  into a power series with respect to  $q$

$$u = u^{(0)} + qu^{(1)} + q^2u^{(2)} + \dots$$

and set  $q = 1$  at the end of calculations. Obviously, if that were all, we would merely obtain a variant of perturbation expansion with artificial “small” parameter introduced by hand, even though  $\mathcal{L}$  need not be contained in  $\mathcal{N}$  and  $u^{(0)}$  need not be a solution to  $\mathcal{L}u = 0$ . However, Liao observed that  $q\mathcal{N}_a(u, t)$  can still be multiplied by a parameter  $\xi$  and function  $h(q)$ , the latter being analytical in  $q$  without changing the solution of (2) for  $q = 1$ . This gives the system:

$$[(1 - q)\mathcal{L}(u(t) - u^{(0)}(t))]_a + \xi q h(q) \mathcal{N}_a(u, t) = 0, \quad (3)$$

The function  $h(q)$  can be chosen in such a way as to, e.g., redefine a “real” expansion parameter (which is not explicitly visible in (1) to make it smaller, while  $\xi$  can be chosen at the end of calculations to improve the overall convergence. In fact the proper choice of the parameter  $\xi$  can and usually does improve the quality of approximate solution quite dramatically, and its apparently insignificant presence facilitated the spectacular success of the method. Further considerable progress has been achieved by Marinca and Herisanu who proposed to optimize the choice of the coefficients in the expansion

$$h(q) = h_0 + h_1q + h_2q^2 + \dots$$

by minimization of residual error. Even with those amendments, the method, in generic case, suffers from the presence of secular terms if applied to resonant systems so that the expansion is not uniformly valid. However, any known mechanism to eliminate the secular terms can be employed to augment the power of Liao’s approach, e.g. the Lindstedt-Poincare method. Here, we will use a variant of the Lindstedt-Poincare method by observing that the

operator  $\mathcal{L}$  can itself be an analytical function of  $q$ . This will allow us to eliminate (at least in the sense of expectation values, please see below) the terms which would correspond to the secular terms in ordinary differential equations.

### 3. Application to stochastic differential equation with third-order nonlinearity

We consider the following non-linear stochastic ordinary differential equation:

$$\frac{d\phi(s)}{ds} = -\alpha\phi(s) - \frac{1}{6}\lambda\phi(s)^3 + f(s), \quad (4)$$

where the stochastic “force”  $f(s)$  is a Gaussian and Markovian and its statistical properties are given by:

$$\langle f(s) \rangle = 0,$$

$$\langle f(s)f(s') \rangle = 2v\delta(s - s'),$$

where  $v$  is a real parameter larger than zero and the sharp brackets denote expectation values.

Physically, the above equation describes heavily overdamped anharmonic oscillations of a Brownian particle. It has been used by Bender and coworkers to introduce strong-coupling expansion in classical statistical mechanics [10]. However, we would like to emphasize here another motivation: Eq.(4) can be obtained as:

$$\frac{d\phi(s)}{ds} = -\frac{dS_E}{d\phi} + f(s), \quad (5)$$

where

$$S_E = \frac{1}{2}\phi a\phi + \frac{1}{4!}\phi^4 \quad (6)$$

is sometimes called the action of a “zero-dimensional” scalar meson field theory with quartic self-interaction. This is because the full four-dimensional action (in imaginary or Euclidean time  $\tau = -it$ ) has the form:

$$S_E = \int d\tau d^3r \left( \frac{1}{2}\Phi(\mathbf{r}, \tau)L_{KG}\Phi(\mathbf{r}, \tau) + \frac{1}{4!}\Phi(\mathbf{r}, t)^4 \right), \quad (7)$$

where  $L_{KG}$  is the linear Klein-Gordon operator written in the imaginary time.

The fact that all statistical properties of the quantized meson field they can be obtained from stationary ( $s \rightarrow \infty$ ) solutions of the system:

$$\frac{d\Phi(s)}{ds} = -\frac{\delta S_E}{\delta \Phi} + f(\mathbf{r}, \tau, s), \quad (8)$$

where  $\delta/\delta\Phi$  is the variational derivative, has been established in [11, 12, 13, 14, 15]. The transition from (7) to (8) is usually called “stochastic quantization”.

By stationary solution of (4) or (8) we mean the solution obtained for large  $s$  so that all the initial correlation have died out and it is sufficient to take into account only the special solution of the inhomogeneous equation.

The fact that (6) is the zero-dimensional caricature of (7) ( $S$  becomes merely a simple function instead being a functional of the field  $\Phi$ ) is evident, and the same is true about (4) and (8). The way any special method is applied to solve (8) is a direct generalization of its application to (4).

Let us now observe that Eq. (4) admits an exact solution in the sense that all moments of  $\phi$  can be easily obtained. Indeed, the corresponding Fokker-Planck equation for the distribution function  $\rho(\phi, s)$  takes the form:

$$\frac{\partial \rho(\phi, s)}{\partial s} = v \frac{\partial^2 \rho(\phi, s)}{\partial \phi^2} + \frac{\partial}{\partial \phi} \left[ \left( a\phi + \frac{1}{6} \lambda \phi^3 \right) \rho(\phi, s) \right], \quad (9)$$

which admits the stationary ( $s$ -independent) solution:

$$\rho = \rho(\phi) = N \exp \left[ -\frac{1}{v} \left( \frac{1}{2} \alpha \phi^2 + \frac{1}{4!} \lambda \phi^4 \right) \right], \quad (10)$$

where  $N$  is a normalization constant obtained from the normalization condition:

$$\int_{-\infty}^{\infty} \rho(\phi) d\phi = 1,$$

so that

$$N = \left[ \sqrt{3} \exp \left( \frac{3\alpha^2}{4\lambda v} \right) \sqrt{\frac{\alpha}{\lambda}} K_{1/4} \left( \frac{3\alpha^2}{4\lambda v} \right) \right]^{-1}$$

This way we obtain, in particular:

$$\langle \phi^2 \rangle_{st} = \frac{e^{-\frac{3\alpha^2}{4\lambda v}} \left( 16\sqrt{3}\sqrt{\lambda v} \Gamma\left(\frac{7}{4}\right) {}_1F_1\left(\frac{3}{4}, \frac{1}{2}, \frac{3\alpha^2}{2\lambda v}\right) - 9\sqrt{2}\alpha \Gamma\left(\frac{1}{4}\right) {}_1F_1\left(\frac{5}{4}, \frac{3}{2}, \frac{3\alpha^2}{2\lambda v}\right) \right)}{32^{3/4} \sqrt[4]{3} \sqrt{\alpha} \lambda^{3/4} K_{\frac{1}{4}}\left(\frac{3\alpha^2}{4\lambda v}\right)},$$

where the braces  $\langle \dots \rangle_{st}$  denote stationary expectation value,  ${}_1F_1$  is the hypergeometric function,  $\Gamma$  is the Euler Gamma function, and  $K_{1/4}$  is the modified Bessel function of the second kind.

Below, we pretend that we do not know the above exact expressions. In order to test whether the homotopy analysis method is suitable for the stochastic differential equations, we shall attempt to solve (4) approximately.

Following Liao, alongside (4) we consider the equation:

$$(1-q)\left(\frac{d}{ds} + \nu(q)\right)\phi(q, s) + qc(q)\left[\frac{d\phi(q, s)}{ds} + \alpha\phi(q, s) + \frac{1}{6}\lambda\phi(q, s)^3\right] = 0, \quad (11)$$

in which we expand  $\phi$ ,  $\nu$ ,  $c$  as:

$$\phi(q, s) = \sum_{n \geq 0} \phi_n q^n,$$

$$\nu(q) = \sum_{n \geq 0} \nu_n q^n,$$

$$c(q) = \sum_{n \geq 0} c_n q^n.$$

However, in order not to overburden the expansions, we will equate all  $c_n$  with non-zero  $n$  to zero. In the end of calculation, we shall set  $q = 1$  and

$$\sum_{n \geq 0} \nu_n q^n|_{q=1} = \alpha. \quad (12)$$

The last condition means that for  $\lambda = 0$ ,  $q = 0$ , we obtain the “unperturbed”, linear problem

$$\frac{d\phi}{ds} = -\alpha\phi + f.$$

Let us notice here that the choice of the constraints in Eq. (12) is only the simplest of many possibilities. In specific cases, other choices may be

preferable. As mentioned before, there is no obvious way to get  $\nu_1$ ,  $\nu_2$ , etc. In the case of deterministic resonance problems we would choose  $\nu_n$ ,  $n \geq 1$  in such a way as to avoid the secular terms. Here, no secular terms appear. Below we propose a scheme to choose  $\nu_n$  which seems to us simple and intuitive. Another choice is discussed in Section 4.

In the zeroth-order we obtain, of course:

$$\left(\frac{d}{ds} + \nu_0\right) \phi_0(s) = f(s), \quad (13)$$

so that

$$\phi_0(s) = \phi_0(0)e^{-\nu_0 s} + \int_0^s e^{-\nu_0(s-s')} f(s') ds' \quad (14)$$

For large  $s$  we have

$$\phi_0(s) = \int_0^s e^{-\nu_0(s-s')} f(s') ds' \quad (15)$$

so that, for large  $s, s'$  the correlation function  $\langle \phi_0(s) \phi_0(s') \rangle$  is given by:

$$\langle \phi_0(s) \phi_0(s') \rangle = \frac{v}{\nu_0} (\exp(-\nu_0|s - s'|) - \exp(-\nu_0(s + s'))), \quad (16)$$

which for  $s = s'$  becomes:

$$\langle \phi_0(s)^2 \rangle = \frac{v}{\nu_0} (1 - \exp(-2\nu_0 s)), \quad (17)$$

and, naturally,  $\langle \phi^2 \rangle_{st} = v/\nu_0$ .

In the first order we need to solve:

$$\left(\frac{d}{ds} + \nu_0\right) \phi_1(s) + (\nu_1 + c_0(a - \nu_0))\phi_0 + \frac{1}{6}c_0\lambda\phi_0(s)^3 = 0. \quad (18)$$

The solution for large  $s$  is given by:

$$\phi_1(s) = - \int_0^s e^{-\nu_0(s-s')} \left[ (\nu_1 + c_0(a - \nu_0))\phi_0(s') + \frac{1}{6}c_0\lambda\phi_0(s')^3 \right] ds'. \quad (19)$$

In order to obtain  $\nu_1$ , we require that:

$$\langle \phi_0 \phi_1 \rangle_{st} = 0.$$

That is, up to the first order in  $q$ , the second moment of  $\phi$  is equal to second order of  $\phi_0$ ,

$$\langle \phi^2 \rangle_{st} = \langle \phi_0^2 \rangle_{st}$$

We have chosen that way to establish  $\nu_1$  because it resembles somewhat the method of consecutive elimination of perturbing terms in the near-identity transformation methods in perturbation theory, please see Chapter 5 of [17]. We find without difficulties that

$$\langle \phi_0 \phi_1 \rangle_{st} = \frac{v}{2\nu_0^2} \left[ \nu_1 + c_0(\alpha - \nu_0) + \frac{1}{2}c_0\lambda\frac{v}{\nu_0} \right],$$

and, therefore,

$$\nu_1 = c_0 \left( \nu_0 - \alpha - \frac{1}{2}c_0\lambda\frac{v}{\nu_0} \right). \quad (20)$$

If we want to risk a seemingly crude approximation and stop the expansion already after the first order, we need to set

$$\nu_0 + \nu_1 = \alpha,$$

which implies the following self-consistent equation for  $\nu_0$ :

$$\nu_0 = \alpha + \frac{1}{2} \frac{c_0}{c_0 + 1} \lambda \frac{v}{\nu_0}. \quad (21)$$

Its solution gives:

$$\nu_0 = \frac{1}{2} \left( \alpha \pm \sqrt{\alpha^2 + 2 \frac{c_0}{c_0 + 1} \lambda v} \right). \quad (22)$$

We need to take the “+” before the square root; otherwise  $\nu_0$  would be smaller than zero.

The question now appears how we can choose  $c_0$ . We could, in principle, follow the route known from the linear delta expansion method [16]. This method is inherently variational in its nature, containing usually an artificial parameter, say,  $\mu$ , which, roughly speaking, plays the role analogous to  $c_0$ . One imposes there the reasonable condition that one of the physical quantities of interest (let us call it  $F$ ) should exhibit “minimal sensitivity” for the change of  $\mu$ . That is, one should have  $\partial F / \partial \mu = 0$ . If we were to



follow that route, our quantity of interest, namely  $\langle \phi^2 \rangle$  would have to have a vanishing derivative with respect to  $c_0$ . However, this cannot be the case, as the above derivative never vanishes. We could either simply proceed to the second order and then check the derivatives, or try to establish another way to find  $c_0$ . A very attractive technique to establish  $c_0$  is that proposed by Marinca and coworkers; it consists in minimizing the “residual error” in  $\phi_1$ . Let us observe here, however, that equally well one can impose conditions based on other error-related criteria, similarly as in the statistical linearization [18, 19, 20, 21]. While we appreciate valuable insights of Marinca and coworkers, in this work we shall use instead the known fact that in the quantum field theory and statistical physics the so-called strong coupling limit is very often available in addition to the weak-coupling one. In our language this means that both limits  $\lambda/\alpha \rightarrow 0$  and  $\alpha/\lambda \rightarrow 0$  are available. In our simple system this is of course the case, and we have in the limit  $\alpha/\lambda \rightarrow 0$ :

$$\langle \phi^2 \rangle_{st} = \frac{2\sqrt{6}\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} \sqrt{\frac{v}{\lambda}}, \quad (23)$$

where  $\Gamma(x)$  is the Euler Gamma function. On the other hand, our approximate  $\langle \phi^2 \rangle$  gives, for  $\alpha = 0$ ,  $\sqrt{2(c_0 + 1)}/\sqrt{c_0}\sqrt{v/l}$ . Hence

$$c_0 = \frac{\Gamma(\frac{1}{4})^2}{12\Gamma(\frac{3}{4})^2 - \Gamma(\frac{1}{4})^2} \approx 2.6965826...$$

A comparison of exact and approximate first-order results with the above  $c_0$  is contained in Fig. 1. The agreement seems quite spectacular if one takes into account the very low order of approximation.

Let us observe that for  $v$  of the order of 1 or larger the exact and first-order approximate solutions are almost indistinguishable.

Predictably, the results for the fourth moments are less impressive, but still remarkable, as shown in Fig. 2

The results for the fourth moments vary from only qualitatively correct for small values of  $v$  to good agreement with exact values for  $v > \alpha$ .

Let us proceed to the second order. Now, we need to solve:

$$\begin{aligned} \left( \frac{d}{ds} + \nu_0 \right) \phi_2(s) &= (\nu_1 + c_0(\alpha - \nu_0))\phi_1(s) + \\ &+ (\nu_2 - c_0\nu_1 + c_0(1 - c_0)(\alpha - \nu_0))\phi_0(s) + \end{aligned} \quad (24)$$

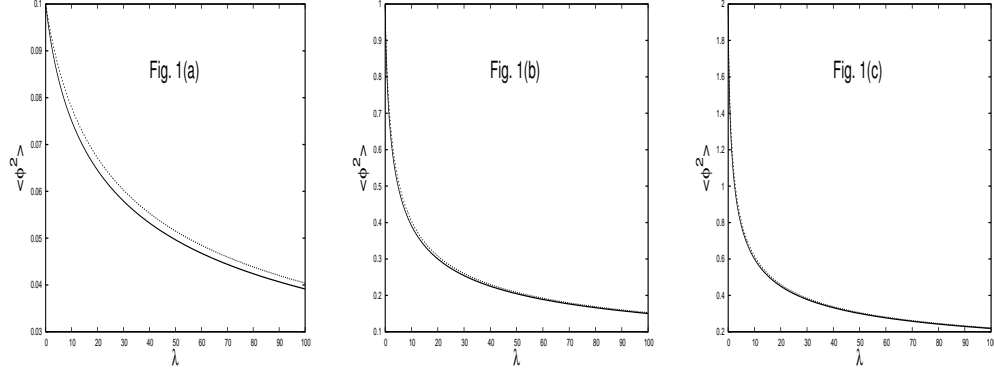


Figure 1: Comparison of exact and first-order approximate results for the dependence of the second moment of the variable  $\phi$  on  $\lambda$  for three different values of  $v$ ; (a)  $v = 0.1$ ; (b)  $v = 1.0$ ; (c)  $v = 2.0$ . The parameter  $\alpha$  is set to 1. The solid line represents exact solution, and the dotted line - the approximate one.

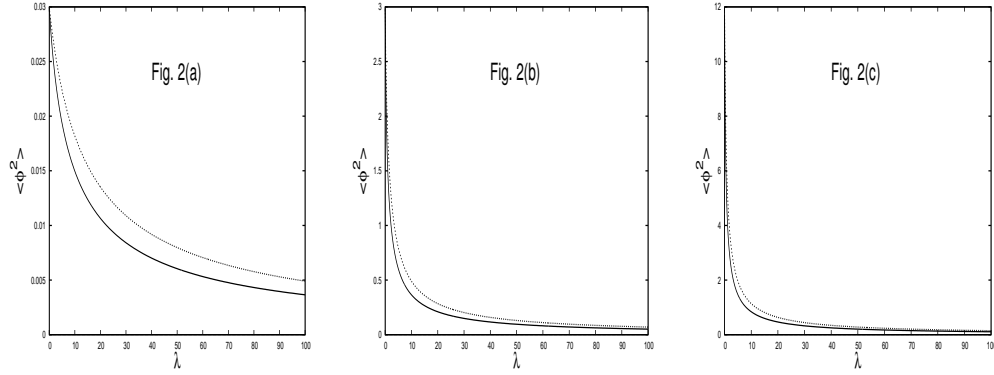


Figure 2: Comparison of exact and first-order approximate results for the dependence of the fourth moment of the variable  $\phi$  on  $\lambda$  for three different values of  $v$ ; (a)  $v = 0.1$ ; (b)  $v = 1.0$ ; (c)  $v = 2.0$ . The parameter  $\alpha$  is set to 1. The solid line represents exact solution, and the dotted line - the approximate ones.

$$+ \frac{1}{6}c_0(1-c_0)\lambda\phi_0(s)^3 + \frac{1}{2}c_0\lambda\phi_0(s)^2\phi_1(s).$$

The solution is again quite simple:

$$\begin{aligned} \phi_2(s) = & - \int_0^s e^{-\nu_0(s-s')} [(\nu_1 + c_0(\alpha - \nu_0))\phi_1(s') + \\ & + (\nu_2 + c_0(\alpha - \nu_0) - c_0(\nu_1 + c_0(\alpha - \nu_0)))\phi_0(s') + \\ & + \frac{1}{6}c_0(1-c_0)\lambda\phi_0(s')^3 + \frac{1}{2}c_0\lambda\phi_0(s')^2\phi_1(s')] ds'. \end{aligned} \quad (25)$$

We require that:

$$\langle \phi^2 \rangle_{st} \approx \langle \phi_0^2 \rangle_{st}, \quad (26)$$

which means that

$$2\langle \phi_0\phi_2 \rangle_{st} + \langle \phi_1^2 \rangle_{st} = 0.$$

The following algebra is simple but somewhat boring because  $\phi_1$  and  $\phi_2$  are no longer Gaussian random variables even though  $\phi_0$  is. It leads to the following self-consistent equations for  $\nu_0$ :

$$\nu_0 = \alpha + \frac{c_0}{2c_0 + 1} \frac{v\lambda}{\nu_0} - \frac{1}{6}c_0^2\lambda^2 \frac{v^2}{\nu_0^3} \quad (27)$$

All higher-order moment can, of course, be expressed with the help of  $v$  and  $\nu_0$  because  $\phi_0$  is Gaussian.

Now, the following difficulty appears. If we try to impose the condition:  $\nu_0 + \nu_1 + \nu_2 = \alpha$ , which is suitable in the second order, we are confronted with the algebraic equation of the fourth degree. While we of course can solve it, it is by no means clear whether the solutions contain a subset of real ones for sufficiently broad range of the parameter  $c_0$ . It is actually *not* the case for Eq. (27). In addition, it is not possible to reach the limit  $\lambda/\alpha \rightarrow \infty$  in the sense that there are no real solutions for  $c_0$ . Therefore, we have adopted the following approach. The function of  $\nu_0$  which stand on the right-hand side of (27) can be understood as consisting just first few terms of a series in powers of  $\nu_0^{-1}$ . We attempted to sum the series using the Pade [1/2] approximant. The result is

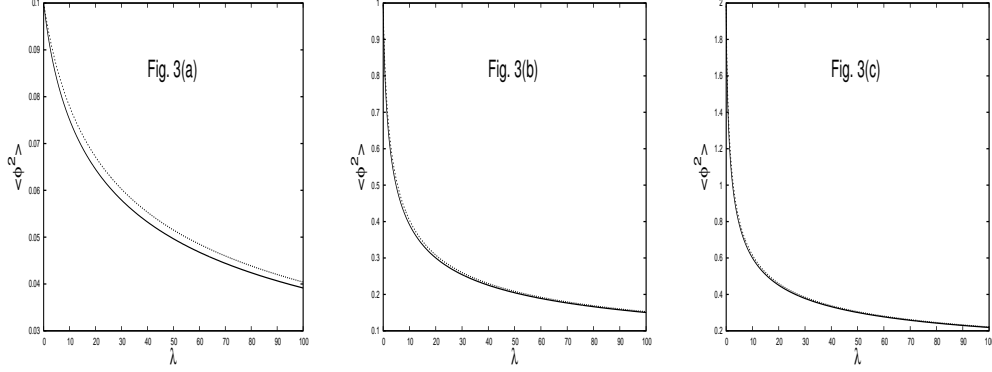


Figure 3: Comparison of exact and first-order approximate results for the dependence of the fourth moment of the variable  $\phi$  on  $\lambda$  for three different values of  $v$ ; (a)  $v = 0.1$ ; (b)  $v = 1.0$ ; (c)  $v = 2.0$ . The parameter  $\alpha$  is set to 1. The solid line represents exact solution, and the dotted line - the approximate one.

$$\nu_0 = \frac{3 c_0^2 l \nu_0 v + (6 \alpha c_0^2 + 3 \alpha c_0) \nu_0^2 + (-4 \alpha^2 c_0^2 - 4 \alpha^2 c_0 - \alpha^2) \nu_0}{(2 c_0^2 + c_0) l v + (6 c_0^2 + 3 c_0) \nu_0^2 + (-4 \alpha c_0^2 - 4 \alpha c_0 - \alpha) \nu_0}. \quad (28)$$

This way we obtain for  $\nu_0$  an algebraic equation of the *third* degree. What is more, one solution is trivial, and it is fairly easy to pick up the proper one from the remaining two (of course, it is to be positive). Then the strong-coupling limit is also achievable. Interestingly, we have found a *negative* value of  $c_0$ , namely,  $-1.762159205046485$ , as that for which the limit of large  $\lambda$  for the second moment is obtained. The resulting dependence of the second moments on  $\lambda$  is shown in Fig. 3.

#### 4. Homotopy analysis method for solution of the Fokker-Planck equation

To every stochastic equations of the form:

$$\frac{d\phi}{ds} = -\alpha\phi - \frac{1}{6}\lambda\phi^3 + f(s)$$

corresponds the following Fokker-Planck equations of the form:

$$\frac{\partial \rho}{\partial s} = \frac{\partial^2 \rho}{\partial \phi^2} + \frac{1}{v} \frac{\partial}{\partial \phi} \left[ \left( \alpha \phi + \frac{1}{6} \lambda \phi^3 \right) \rho \right] \quad (29)$$

where  $\rho$  is the distribution function of the random variable  $\phi$ . Here, we are interested in the stationary ( $s$ -independent) solution of the Fokker-Planck equation. It is, of course, immediately available and given by (10). In order to check validity of the homotopy analysis method we will apply it the Fokker-Planck equations and determine the approximate solution.

Alongside Eq. (29) with left-hand side equal to zero we consider the equation:

$$\begin{aligned} (1-q) \left[ \frac{\partial^2 \rho}{\partial \phi^2} + \frac{1}{v} \frac{\partial \rho}{\partial \phi} [(\nu \phi) \rho] \right] + \\ + q c_0 \left[ \frac{\partial^2 \rho}{\partial \phi^2} + \frac{1}{v} \frac{\partial}{\partial \phi} \left( \left( \alpha \phi + \frac{1}{6} \lambda \phi^3 \right) \rho \right) \right] = 0. \end{aligned} \quad (30)$$

We look for the solution of (30) in terms of the expansion:

$$\rho = \rho_0 + q \rho_1 + q^2 \rho_2 + \dots$$

with analogous expansion of  $\nu$ ,  $\nu = \nu_0 + q \nu_1 + q^2 \nu_2 + \dots$

In the zeroth-order with respect to  $q$  we obtain:

$$\frac{\partial^2 \rho_0}{\partial \phi^2} + \frac{1}{v} \frac{\partial}{\partial \phi} (\nu_0 \phi \rho_0) = 0. \quad (31)$$

The solution which vanishes at  $\pm \infty$  is, obviously,

$$\rho_0 = N \exp\left(-\frac{1}{2} \frac{\nu_0}{v} \phi^2\right), \quad (32)$$

where  $N$  is a normalization constant. The differential equation for  $\rho_1$  is

$$\frac{d^2 \rho_1}{d\phi^2} + \frac{1}{v} \frac{d}{d\phi} (\nu_0 \phi \rho_1) = -\frac{1}{v} \frac{d}{d\phi} \left( ((c_0(\alpha - \nu_0) + \nu_1) \phi + \frac{1}{6} c_0 \lambda \phi^3) \rho_0 \right). \quad (33)$$

Integrating once and requiring that the integration constant be zero (otherwise the solution would not be square-integrable) leads to the simple expression:

$$\frac{d\rho_1}{d\phi} + \frac{1}{v}(\nu_0\phi\rho_1) = \frac{-1}{v} \left[ ((c_0(\alpha - \nu_0) + \nu_1)\phi + \frac{1}{6}\lambda c_0\phi^3)\rho_0 \right] = 0. \quad (34)$$

Let us now represent  $\rho_1$  as the product of  $\rho_0$  and certain  $\bar{\rho}_1$ ,  $\rho_1 = \rho_0\bar{\rho}_1$ . Then for  $\bar{\rho}_1$  we obtain simply:

$$\frac{d\bar{\rho}_1}{d\phi} = -\frac{1}{v} \left[ (c_0(\alpha - \nu_0) + \nu_1)\phi + \frac{1}{6}c_0\lambda\phi^3 \right]. \quad (35)$$

Thus,

$$\bar{\rho}_1 = \frac{-1}{v} \left[ \frac{1}{2}(c_0(\alpha - \nu_0) + \nu_1)\phi^2 + \frac{1}{24}c_0\lambda\phi^4 \right]. \quad (36)$$

If we want to stop the expansion already after the calculations of the first-order solution, we need a relation between  $\nu_1$  and  $\nu_0$ . We impose the condition that, up to the first order, the second moment of  $\phi$  is given solely by  $\rho_0$ , that is:

$$\int_{-\infty}^{\infty} \phi^2 \rho_1(\phi) = 0. \quad (37)$$

This leads to the following relation between  $\nu_1$  and  $\nu_0$ :

$$\nu_1 + c_0(\alpha - \nu_0) = -\frac{5}{12}c_0\lambda\frac{v}{\nu_0}, \quad (38)$$

or, taking into account that  $\nu_0 + \nu_1 = \alpha$ , to

$$\nu_0 = \frac{1}{2} \left( \alpha \pm \sqrt{\alpha^2 + \frac{5}{3}\frac{c_0}{c_0}\lambda v} \right). \quad (39)$$

On the other hand, if we require that  $\phi$  and  $d\bar{\rho}_1/d\phi$  are orthogonal in the sense of the scalar product

$$((.), (.)) = \int_{-\infty}^{\infty} (.) (.) \rho_0 d\phi,$$

then  $\nu_0$  is given by the same formula as in the previous Section:

$$\nu_0 = \frac{1}{2} \left( \alpha \pm \sqrt{\alpha^2 + 2\frac{c_0}{c_0 + 1}\lambda v} \right).$$

Remarkably, the two first-order expressions of  $\nu_0$  are very similar but not identical.

For  $\rho_2$  we obtain the following equation:

$$\begin{aligned} & \frac{d^2 \rho_2}{d\phi^2} + \frac{1}{v} \frac{d}{d\phi} (\nu_0 \phi \rho_2) + \frac{1}{v} \frac{d}{d\phi} (\nu_1 \phi \rho_1) + \frac{1}{v} \frac{d}{d\phi} (\nu_2 \phi \rho_0) + \\ & + c_0 \left[ \frac{d^2 \rho_0}{d\phi^2} + \frac{1}{v} \frac{d}{d\phi} \left( \left( \alpha \phi + \frac{\lambda}{6} \phi^3 \right) \rho_0 \right) \right] + \\ & + c_0 \left[ \frac{d^2 \rho_1}{d\phi^2} + \frac{1}{v} \frac{d}{d\phi} \left( \left( \alpha \phi + \frac{\lambda}{6} \phi^3 \right) \rho_1 \right) \right] = 0 \end{aligned} \quad (40)$$

Simple manipulations and one integration over  $\phi$  leads to the following expression:

$$\begin{aligned} \frac{d\rho_2}{d\phi} &= -\frac{1}{v} (\nu_0 \phi \rho_2) - \frac{1}{v} [(\nu_1 + \nu_0(\alpha - \nu_0))\phi + \frac{1}{6} c_0 \lambda \phi^3] \rho_1 - \\ &- \frac{1}{v} [(\nu_2 + c_0(\alpha - \nu_0) - c_0(\nu_1 + c_0(\alpha - \nu_0)))\phi \rho_0 + \frac{1}{6} (c_0 - c_0^2) \lambda \phi^3 \rho_0]. \end{aligned} \quad (41)$$

On writing  $\rho_2 = \rho_0 \bar{\rho}_2$  we obtain

$$\begin{aligned} \frac{d\bar{\rho}_2}{d\phi} &= -\frac{1}{v} [(\nu_1 + \nu_0(\alpha - \nu_0))\phi + \frac{1}{6} c_0 \lambda \phi^3] \bar{\rho}_1 - \\ &- \frac{1}{v} [(\nu_2 + c_0(\alpha - \nu_0) - c_0(\nu_1 + c_0(\alpha - \nu_0)))\phi + \frac{1}{6} (c_0 - c_0^2) \lambda \phi^3]. \end{aligned} \quad (42)$$

If we now require that  $\phi$  is orthogonal to both  $d\bar{\rho}_1/d\phi$  and  $d\bar{\rho}_2/d\phi$  with respect to the scalar product defined above, we obtain the relation

$$\nu_2 + c_0(\alpha - \nu_0) + \frac{1}{2} c_0 \lambda \frac{v}{\nu_0} = \frac{1}{6} c_0^2 \lambda^2 \frac{v^2}{\nu_0^3}.$$

What is more, if we decide to break the expansion in terms of  $q$  at the second order so that  $\nu_0 + \nu_1 + \nu_2 = \alpha$ , we get the implicit (self-consistent) expression for  $\nu_0$ :

$$\nu_0 = \alpha + \frac{c_0}{2c_0 + 1} \lambda \frac{v}{\nu_0} - \frac{1}{6} \frac{c_0^2}{2c_0 + 1} \frac{v^2}{\nu_0^3} \quad (43)$$

which is identical with that formerly obtained directly from the Langevin equation.

The solution to Eq. (42) can be written as:

$$\begin{aligned} \bar{\rho}_2 = & \frac{1}{v^2} \left[ \frac{1}{8} (\nu_1 + c_0(\alpha - \nu_0))^2 \phi^4 + \frac{1}{48} c_0 \lambda (\nu_1 + c_0(\alpha - \nu_0)) \phi^6 + \right. \\ & \left. + \frac{1}{1152} c_0^2 \lambda^2 \phi^8 \right] - \frac{1}{v} \left[ \frac{1}{2} (\nu_2 + c_0(\alpha - \nu_0) - c_0(\nu_1 + c_0(\alpha - \nu_0))) \phi^2 \right] + \frac{1}{24} (c_0 - c_0^2) \lambda \phi^4 \end{aligned} \quad (44)$$

Let us now require that only  $\rho_0$  be needed to obtain the second moment of  $\phi$ , so that both equations

$$\int_{-\infty}^{\infty} \phi^2 \rho_1 d\phi = 0,$$

and

$$\int_{-\infty}^{\infty} \phi^2 \rho_2 d\phi = 0$$

hold. In that case we obtain the following relation between  $\nu_2$  and  $\nu_0$ :

$$\nu_2 + c_0(\alpha - \nu_0) + \frac{5}{12} c_0 \lambda \frac{v}{\nu_0} = \frac{5}{32} c_0^2 \lambda^2 \frac{v^2}{\nu_0^3},$$

and if we want to finish our expansion at the second order,  $\nu_0$  has to satisfy:

$$\nu_0 = \alpha + \frac{5}{6} \frac{c_0}{2c_0 + 1} \frac{v}{\nu_0} - \frac{5}{32} \frac{c_0^2}{2c_0 + 1} \lambda^2 \frac{v^2}{\nu_0^3} \quad (45)$$

Let us notice that it is only the numerical coefficients which make the Eq. (45) slightly different from (27). When we applied the same procedure, i.e. Pade resummation, to the right-hand side of (45), solve for  $\nu_0$  and take the limit of  $\lambda \rightarrow \infty$  to obtain  $c_0$  the resulting dependence of the second moment of  $\phi$  on  $\lambda$  differs only very slightly from that obtained from (27).

## 5. Concluding remarks

In this paper we have applied the homotopy analysis method to a nonlinear stochastic differential equation with Gaussian-Markovian stochastic force as well as to the corresponding Fokker-Planck equation. The value of an artificial parameter the presence of which is characteristic for the method has been



fixed by taking the limit of strong nonlinearity. Approximate solutions have been obtained and compared with the exact ones. It has been demonstrated that the second moments of the dependent stochastic variables as given by the homotopy analysis method agree very remarkably with exact moments. Broad perspectives of applications for the method seem to be open; for, despite the simplicity of the considered model, it possesses some characteristics of the stochastic differential equations of statistical mechanics and quantum field theory in the so-called stochastic quantization.

Work is in progress on application of both those techniques in the physics of cold atomic gases, and quantum optics.

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